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2004 J. Phys. A: Math. Gen. 37 8657

(http://iopscience.iop.org/0305-4470/37/36/006)

J. Phys. A: Math. Gen. **37** (2004) 8657–8674 PII: S0305-4470(04)81585-3

Eigenvalue accumulation for Dirac operators with spherically symmetric potential

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Received 3 June 2004
Published 24 August 2004
Online at stacks.iop.org/JPhysA/37/8657
doi:10.1088/0305-4470/37/36/006

Abstract
We consider Dirac operators H in $\mathbb{R}^3$ with spherically symmetric potentials. The main result is a criterion for eigenvalue accumulation and non-accumulation at the endpoints -1 and 1 of the essential spectrum under rather weak assumptions on the potential. This result is proved by showing an analogous criterion for the associated radial Dirac operators H_κ and by proving that for $|\kappa|$ sufficiently large, each H_κ does not have any eigenvalues in the interval $(-1, 0]$ and $[0, 1)$, respectively, of the gap $(-1, 1)$ of the essential spectrum. For the latter, properties of solutions of certain Riccati differential equations depending on the parameter κ and the spectral parameter are used.

PACS numbers: 03.65.Pm, 02.30.Hq, 02.30.Tb
Mathematics Subject Classification: 81Q10, 34L40, 35Q40

1. Introduction

For the radial Dirac operators H_κ, $\kappa \in \mathbb{Z}\backslash\{0\}$, associated with the Dirac operator H in $L^2(\mathbb{R}^3)^4$ with a spherically symmetric potential V, criteria for eigenvalue accumulation and non-accumulation at the endpoints -1 and 1 of the essential spectrum are well known (see [4, 10]). However, these criteria do not allow to draw conclusions for the Dirac operator H itself, which is the direct sum of the radial Dirac operators H_κ, $\kappa \in \mathbb{Z}\backslash\{0\}$: even if an endpoint is no accumulation point for any H_κ, it could well be an accumulation point for H.

In this paper, we solve the problem of eigenvalue accumulation at -1 and 1 for the Dirac operator H. To this end, we show that for $|\kappa|$ sufficiently large, each H_κ does not have any eigenvalues in the interval $(-1, 0]$ and $[0, 1)$, respectively. For the proof of this fact we develop a theory for Riccati differential equations depending on two parameters (κ and the spectral parameter), which is also of independent interest. As a second ingredient, we study principal solutions of Dirac systems depending on parameters and establish comparison theorems for them.

The paper is organized as follows: In section 2, we study families of Riccati differential equations of the form

$$z'(x) = a(x,\lambda)z(x)^2 + 2\kappa b(x,\lambda)z(x) + c(x,\lambda), \qquad x \in \Omega,$$

on an interval $\Omega = (0, \omega]$ where $\kappa \in I := \mathbb{R}\backslash(-\beta, \beta)$ with some constant $\beta > 0$ and λ is a parameter varying in some interval $\Lambda \subset \mathbb{R}$, and we investigate the behaviour of their solutions for $\kappa \to \pm\infty$. For this purpose, we reduce the Riccati equation to an integral equation and we apply a technique related to the method used in [1] for the uniform asymptotic integration of linear differential systems.

In section 3, these results are used for a detailed analysis of fundamental matrices of Dirac systems

$$Jy'(x) + \begin{pmatrix} a(x,\lambda) & \kappa b(x,\lambda) \\ \kappa b(x,\lambda) & c(x,\lambda) \end{pmatrix} y(x) = 0, \qquad x \in \Omega, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

depending on κ and λ. The eigenvalue equation for each radial Dirac operator is a special case of such a system for which $b(x,\lambda) = 1/x$, $a(x,\lambda) = V(x)-1-\lambda$ and $c(x,\lambda) = V(x)+1-\lambda$. Section 4 contains a comparison theorem for Dirac systems of the general type above.

In section 5, we study the Dirac operator H in $\mathrm{L}^2(\mathbb{R}^3)^4$ with spherically symmetric potential $V \in \mathrm{L}^\infty_{\mathrm{loc}}(0,\infty)$ such that $\lim_{x\to\infty} V(x) = 0$ and $\limsup_{x\to 0} |xV(x)| < \frac{1}{2}\sqrt{3}$. The operator H can be decomposed as a direct sum of radial Dirac operators

$$H = -\mathrm{i}\alpha\cdot\nabla + \beta + V(|x|)I \cong \bigoplus_{\kappa\in\mathbb{Z}\backslash\{0\}} \bigoplus_{\ell=1}^{\ell(\kappa)} H_\kappa$$

where

$$H_\kappa y(x) = Jy'(x) + \begin{pmatrix} -1 + V(x) & \frac{\kappa}{x} \\ \frac{\kappa}{x} & 1 + V(x) \end{pmatrix} y(x), \qquad x \in (0,\infty).$$

For the operator H and the radial Dirac operators H_κ the essential spectrum is well known to be $\mathbb{R}\backslash(-1,1)$.

For the radial Dirac operators H_κ, we show that the eigenvalues in $(-1,1)$ accumulate, e.g., at 1 if $\limsup_{x\to\infty} x^2 V(x) < -\frac{1}{8}(2\kappa+1)^2$ and they do not accumulate at 1 if $\liminf_{x\to\infty} x^2 V(x) > -\frac{1}{8}(2\kappa+1)^2$. This is a generalization of a result in [10] which was proved by applying the Levinson theorem (see [2]) and required in addition that $\int_0^1 \left|V(x) - \frac{\rho}{x}\right| \mathrm{d}x < \infty$ with some $\rho \in \left[0, \frac{1}{2}\sqrt{3}\right)$.

The key point of this paper is theorem 5.1 showing that $\liminf_{x\to\infty} x^2 V(x) > -\infty$ already implies that H_κ has *no* eigenvalues in $[0,1)$ for sufficiently large $|\kappa|$. For the proof, the results of section 3 are used to show that a necessary interface condition for solutions of the eigenvalue equation in $(0,\omega]$ and $[\omega,\infty)$ cannot be satisfied.

Finally, theorem 5.1 and the eigenvalue accumulation criterion for the radial Dirac operators together show that the eigenvalues of the Dirac operator in $(-1,1)$

accumulate at 1 if $\limsup_{x\to\infty} x^2 V(x) < -\frac{1}{8}$,

do not accumulate at 1 if $\liminf_{x\to\infty} x^2 V(x) > -\frac{1}{8}$.

An analogous result holds for the other endpoint -1.

2. Riccati equations depending on some parameter

In this section, we study a family of Riccati differential equations

$$z'(x) = a(x,\lambda)z(x)^2 + 2\kappa b(x,\lambda)z(x) + c(x,\lambda), \qquad x \in \Omega, \tag{2.1}$$

on an interval $\Omega = (0,\omega]$, $0 < \omega < \infty$, where $\kappa \in I := \mathbb{R}\backslash(-\beta,\beta)$ with some constant $\beta > 0$ and λ is a parameter varying in some interval $\Lambda \subset \mathbb{R}$. We assume that the coefficients $a, b, c : \Omega \times \Lambda \longrightarrow \mathbb{R}$ satisfy the following conditions:

(i) The functions $b(\cdot,\lambda)$ are locally integrable on Ω for all $\lambda \in \Lambda$, the functions $b(x,\cdot)$ are continuous on Λ for all $x \in \Omega$, and there exists a locally integrable function $B : \Omega \longrightarrow \mathbb{R}$ such that $0 < b(x,\lambda) \leqslant B(x)$ for all $(x,\lambda) \in \Omega \times \Lambda$ and a point $\xi \in (0,\omega)$ such that

$$\delta := \inf_{\lambda\in\Lambda} \int_\xi^\omega b(t,\lambda)\,\mathrm{d}t > 0. \tag{2.2}$$

(ii) The functions $a(\cdot,\lambda), c(\cdot,\lambda)$ are measurable on Ω for all $\lambda \in \Lambda$, the functions $a(x,\cdot), c(x,\cdot)$ are continuous on Λ for all $x \in \Omega$,

$$\alpha := \sup_{(x,\lambda)\in\Omega\times\Lambda} \frac{|a(x,\lambda)|}{b(x,\lambda)} < \infty, \qquad \gamma := \sup_{(x,\lambda)\in\Omega\times\Lambda} \frac{|c(x,\lambda)|}{b(x,\lambda)} < \infty, \tag{2.3}$$

and α, γ satisfy the inequality

$$\alpha\gamma < \beta^2.$$

For a fixed $(\kappa,\lambda) \in I \times \Lambda$, a function $z : \Omega \longrightarrow \mathbb{R}$ is called a *solution* of (2.1) if z is absolutely continuous and (2.1) holds almost everywhere in Ω. Here we are interested in continuous and bounded solutions of (2.1).

Theorem 2.1. *If the coefficients of (2.1) satisfy the conditions (i) and (ii), then there exist solutions $z_\kappa(\cdot,\lambda)$ of the differential equation (2.1) for all $(\kappa,\lambda) \in I \times \Lambda$ such that z_κ is continuous on $\Omega \times \Lambda$, bounded by*

$$\mu_\kappa := \frac{\gamma}{|\kappa| + \sqrt{\kappa^2 - \alpha\gamma}}$$

for all $\kappa \in I$ and has the following properties: If $\kappa \geqslant \beta$, then

$$z_\kappa(\omega,\cdot) \equiv 0 \quad \textit{on } \Lambda.$$

If $\kappa < 0$, then

$$\liminf_{\kappa\to-\infty} \inf_{\lambda\in\Lambda} |\kappa| z_\kappa(\omega,\lambda) \geqslant \tfrac{1}{2}\gamma_* \qquad \textit{if} \quad \gamma_* := \inf_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{c(x,\lambda)}{b(x,\lambda)} > 0,$$

$$\limsup_{\kappa\to-\infty} \sup_{\lambda\in\Lambda} |\kappa| z_\kappa(\omega,\lambda) \leqslant \tfrac{1}{2}\gamma^* \qquad \textit{if} \quad \gamma^* := \sup_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{c(x,\lambda)}{b(x,\lambda)} < 0.$$

Proof. First, we define

$$\phi(x,\lambda) := -2\int_x^\omega b(t,\lambda)\,\mathrm{d}t, \qquad (x,\lambda) \in \Omega \times \Lambda.$$

Since $b(x,\cdot)$ is continuous on Λ for all $x \in \Omega$, $|b(\cdot,\lambda)|$ is bounded by B for all $\lambda \in \Lambda$, and B is locally integrable on Ω, Lebesgue's dominated convergence theorem implies that ϕ is continuous on $\Omega \times \Lambda$. In addition, $\phi(\cdot,\lambda)$ is a non-positive monotonically increasing function on Ω for all $\lambda \in \Lambda$ with $\frac{\partial}{\partial x}\phi(x,\lambda) = 2b(x,\lambda)$ and $\phi(\omega,\lambda) = 0$. For a fixed index $\kappa \in I$, let $\mathcal{E}_\kappa$ be the space of continuous functions $g : \Omega \times \Lambda \longrightarrow [-\mu_\kappa, \mu_\kappa]$. If we introduce the Chebyshev metric

$$d_\kappa(f, g) := \sup_{(x,\lambda)\in\Omega\times\Lambda} |f(x, \lambda) - g(x, \lambda)|, \qquad f, g \in \mathcal{E}_\kappa,$$

then $(\mathcal{E}_\kappa, d_\kappa)$ is a complete metric space. Further, if $\kappa \geqslant \beta$, let

$$(\mathcal{F}_\kappa g)(x, \lambda) := -\mathrm{e}^{\kappa\phi(x,\lambda)} \int_x^\omega [a(t, \lambda)g(t, \lambda)^2 + c(t, \lambda)]\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\, \mathrm{d}t,$$

and if $\kappa \leqslant -\beta$, define

$$(\mathcal{F}_\kappa g)(x, \lambda) := \mathrm{e}^{\kappa\phi(x,\lambda)} \int_0^x [a(t, \lambda)g(t, \lambda)^2 + c(t, \lambda)]\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\, \mathrm{d}t$$

for all $(x, \lambda) \in \Omega \times \Lambda$ and $g \in \mathcal{E}_\kappa$. From (2.3), $|g(t, \lambda)| \leqslant \mu_\kappa$ and $\alpha\mu_\kappa^2 + \gamma = 2|\kappa|\mu_\kappa$, it follows that

$$\begin{aligned}|a(t, \lambda)g(t, \lambda)^2 + c(t, \lambda)|\, \mathrm{e}^{-\kappa\phi(t,\lambda)} &\leqslant 2|\kappa|\mu_\kappa b(t, \lambda)\, \mathrm{e}^{-\kappa\phi(t,\lambda)} \\ &= \operatorname{sign}(-\kappa)\mu_\kappa \frac{\partial}{\partial t}\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\end{aligned}$$

for all $(t, \lambda) \in \Omega \times \Lambda$ and $g \in \mathcal{E}_\kappa$. Hence, if $\kappa \geqslant \beta$, we have

$$|(\mathcal{F}_\kappa g)(x, \lambda)| \leqslant -\mu_\kappa\, \mathrm{e}^{\kappa\phi(x,\lambda)} \int_x^\omega \frac{\partial}{\partial t}\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\, \mathrm{d}t = \mu_\kappa(1 - \mathrm{e}^{\kappa\phi(x,\lambda)}) \leqslant \mu_\kappa$$

since $\phi(\omega, \lambda) = 0$ and $0 \leqslant \mathrm{e}^{\kappa\phi(x,\lambda)} \leqslant 1$. Further, if $\kappa \leqslant -\beta$, we get

$$|(\mathcal{F}_\kappa g)(x, \lambda)| \leqslant \mu_\kappa\, \mathrm{e}^{\kappa\phi(x,\lambda)} \int_0^x \frac{\partial}{\partial t}\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\, \mathrm{d}t = \mu_\kappa(1 - \psi(\lambda)\, \mathrm{e}^{\kappa\phi(x,\lambda)}) \leqslant \mu_\kappa$$

where $\psi(\lambda) := \lim_{t\to 0} \mathrm{e}^{-\kappa\phi(t,\lambda)}$ (this limit exists since $-\kappa\phi(\cdot, \lambda)$ is a non-positive increasing function) and

$$0 \leqslant \psi(\lambda)\, \mathrm{e}^{\kappa\phi(x,\lambda)} = \lim_{t\to 0} \exp\left(2\kappa \int_t^x b(s, \lambda)\, \mathrm{d}s\right) \leqslant 1.$$

These estimates imply that $\mathcal{F}_\kappa g$ is well defined for all $g \in \mathcal{E}_\kappa$ and that $|(\mathcal{F}_\kappa g)(x, \lambda)|$ is bounded by μ_κ for all $(x, \lambda) \in \Omega \times \Lambda$. Moreover, by Lebesgue's dominated convergence theorem, $\mathcal{F}_\kappa g$ is continuous on $\Omega \times \Lambda$. Hence, $\mathcal{F}_\kappa$ maps $\mathcal{E}_\kappa$ into itself. In the following, we prove that $\mathcal{F}_\kappa : \mathcal{E}_\kappa \longrightarrow \mathcal{E}_\kappa$ is a contraction. For this let $g, h \in \mathcal{E}_\kappa$. From (2.3) and $|g(t, \lambda)^2 - h(t, \lambda)^2| \leqslant 2\mu_\kappa d_\kappa(g, h)$ we obtain that

$$\begin{aligned}|a(t, \lambda)(g(t, \lambda)^2 - h(t, \lambda)^2)|\, \mathrm{e}^{-\kappa\phi(t,\lambda)} &\leqslant 2\alpha\mu_\kappa d_\kappa(g, h)b(t, \lambda)\, \mathrm{e}^{-\kappa\phi(t,\lambda)} \\ &= \operatorname{sign}(-\kappa)q_\kappa d_\kappa(g, h)\frac{\partial}{\partial t}\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\end{aligned}$$

for all $(x, \lambda) \in \Omega \times \Lambda$ where

$$0 \leqslant q_\kappa := \frac{\alpha\mu_\kappa}{|\kappa|} = 1 - \frac{\sqrt{\kappa^2 - \alpha\gamma}}{|\kappa|} < 1.$$

Hence, if $\kappa \geqslant \beta$, then

$$\begin{aligned}|(\mathcal{F}_\kappa g)(x, \lambda) - (\mathcal{F}_\kappa h)(x, \lambda)| &\leqslant -q_\kappa d_\kappa(g, h)\, \mathrm{e}^{\kappa\phi(x,\lambda)} \int_x^\omega \frac{\partial}{\partial t}\, \mathrm{e}^{-\kappa\phi(t,\lambda)}\, \mathrm{d}t \\ &= q_\kappa d_\kappa(g, h)(1 - \mathrm{e}^{\kappa\phi(x,\lambda)}) \leqslant q_\kappa d_\kappa(g, h),\end{aligned}$$

and if $\kappa \leqslant -\beta$, it follows that

$$|(\mathcal{F}_\kappa g)(x,\lambda) - (\mathcal{F}_\kappa h)(x,\lambda)| \leqslant q_\kappa d_\kappa(g,h)\,\mathrm{e}^{\kappa\phi(x,\lambda)} \int_0^x \frac{\partial}{\partial t}\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t$$
$$= q_\kappa d_\kappa(g,h)(1-\psi(\lambda)\,\mathrm{e}^{\kappa\phi(x,\lambda)}) \leqslant q_\kappa d_\kappa(g,h)$$

for all $(x,\lambda) \in \Omega \times \Lambda$. Thus $\mathcal{F}_\kappa$ is a contraction on $\mathcal{E}_\kappa$. Now Banach's fixed point theorem implies that there exists a function $z_\kappa \in \mathcal{E}_\kappa$ which satisfies $z_\kappa = \mathcal{F}_\kappa z_\kappa$, and it is easy to verify that $z_\kappa(\cdot,\lambda)$ is also a solution of the differential equation (2.1) for all $(\kappa,\lambda) \in I \times \Lambda$. Additionally, $z_\kappa(\omega,\cdot) \equiv 0$ on Λ if $\kappa \geqslant \beta$.

In order to prove the first of the last two estimates in theorem 2.1, assume that $\gamma_* > 0$. Since $z_\kappa = \mathcal{F}_\kappa z_\kappa$ and $\phi(\omega,\lambda) = 0$, we obtain

$$z_\kappa(\omega,\lambda) = \int_0^\omega c(t,\lambda)\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t + \int_0^\omega a(t,\lambda) z_\kappa(t,\lambda)^2\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t$$

for all $(\kappa,\lambda) \in (-\infty,-\beta] \times \Lambda$. From (2.2), (2.3) and the assumption that $\gamma_* > 0$, it follows that

$$2|\kappa| z_\kappa(\omega,\lambda) \geqslant \int_\xi^\omega 2\gamma_*|\kappa| b(t,\lambda)\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t - \int_0^\xi 2\gamma|\kappa| b(t,\lambda)\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t$$
$$- \int_0^\omega 2\alpha\mu_\kappa^2|\kappa| b(t,\lambda)\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t$$
$$= \gamma_* \int_\xi^\omega \frac{\partial}{\partial t}\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t - \gamma \int_0^\xi \frac{\partial}{\partial t}\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t - \alpha\mu_\kappa^2 \int_0^\omega \frac{\partial}{\partial t}\,\mathrm{e}^{-\kappa\phi(t,\lambda)}\,\mathrm{d}t$$

and further, observing that $\phi(\omega,\lambda) = 0$,

$$2|\kappa| z_\kappa(\omega,\lambda) \geqslant \gamma_* - (\gamma_*+\gamma)\,\mathrm{e}^{-\kappa\phi(\xi,\lambda)} + \left(\gamma + \alpha\mu_\kappa^2\right)\psi(\lambda) - \alpha\mu_\kappa^2$$
$$\geqslant \gamma_* - (\gamma_*+\gamma)\,\mathrm{e}^{-\kappa\phi(\xi,\lambda)} - \alpha\mu_\kappa^2$$
$$\geqslant \gamma_* - (\gamma_*+\gamma)\,\mathrm{e}^{2\kappa\delta} - \alpha\mu_\kappa^2$$

for all $\kappa \in (-\infty,-\beta]$. Since $\lim_{\kappa\to-\infty} \mu_\kappa = 0$, we obtain

$$\liminf_{\kappa\to-\infty} \inf_{\lambda\in\Lambda} |\kappa| z_\kappa(\omega,\lambda) \geqslant \tfrac{1}{2}\gamma_*.$$

The proof of the last estimate is analogous. □

3. Dirac systems depending on some parameter

In the following, we consider the family of Dirac systems

$$Jy'(x) + Q_\kappa(x,\lambda)y(x) = 0, \qquad x \in \Omega, \tag{3.1}$$

on the interval $\Omega = (0,\omega]$, $0 < \omega < \infty$, where $\kappa \in I := \mathbb{R}\backslash(-\beta,\beta)$ with some $\beta > 0$, λ is a parameter varying in some interval $\Lambda \subset \mathbb{R}$, and

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q_\kappa(x,\lambda) := \begin{pmatrix} a(x,\lambda) & \kappa b(x,\lambda) \\ \kappa b(x,\lambda) & c(x,\lambda) \end{pmatrix}, \qquad (x,\lambda) \in \Omega \times \Lambda. \tag{3.2}$$

We assume that the coefficients $a, b, c : \Omega \times \Lambda \longrightarrow \mathbb{R}$ of Q_κ in (3.2) satisfy the conditions (i) and (ii) of the previous section.

For a fixed $(\kappa,\lambda) \in I \times \Lambda$, a function $y : \Omega \longrightarrow \mathbb{R}^2$ is called a *solution* of (3.1), if (every component of) y is absolutely continuous and (3.1) holds almost everywhere in Ω. Further, a *fundamental matrix* of (3.1) is a function $Y : \Omega \longrightarrow \mathrm{M}_2(\mathbb{R})$ (the set of all 2×2 matrices over $\mathbb{R}$) with the property that every solution y of (3.1) can be expressed as $y(x) = Y(x)c$, $x \in \Omega$, with some vector $c \in \mathbb{R}^2$.

Theorem 3.1. *If the conditions (i) and (ii) are satisfied, then, for all* $(\kappa, \lambda) \in I \times \Lambda$, *there exists a fundamental matrix*

$$Y_\kappa(x, \lambda) = \begin{pmatrix} u_\kappa^{(1)}(x, \lambda) & u_\kappa^{(2)}(x, \lambda) \\ v_\kappa^{(1)}(x, \lambda) & v_\kappa^{(2)}(x, \lambda) \end{pmatrix}, \qquad x \in \Omega, \tag{3.3}$$

of (3.1) with the following properties:

(a) The functions $u_\kappa^{(1)}, v_\kappa^{(1)}$ *are continuous on* $\Omega \times \Lambda$, $u_\kappa^{(1)}(x, \lambda) > 0$ *and*

$$u_\kappa^{(1)}(x, \lambda) \begin{cases} \leqslant \exp\left(-\sqrt{\kappa^2 - \alpha\gamma} \int_x^\omega b(t, \lambda)\, \mathrm{d}t\right) & \text{if} \quad \kappa \in [\beta, \infty), \\ \geqslant \exp\left(\sqrt{\kappa^2 - \alpha\gamma} \int_x^\omega b(t, \lambda)\, \mathrm{d}t\right) & \text{if} \quad \kappa \in (-\infty, -\beta] \end{cases}$$

for all $(x, \kappa, \lambda) \in \Omega \times I \times \Lambda$. *Moreover,*

$$\sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{v_\kappa^{(1)}(x, \lambda)}{u_\kappa^{(1)}(x, \lambda)} \right| \leqslant \alpha$$

for all $\kappa \in I$, $v_\kappa^{(1)}(\omega, \cdot) \equiv 0$ *on* Λ *for all* $\kappa \in (-\infty, -\beta]$, *and*

$$\limsup_{\kappa\to+\infty} \sup_{\lambda\in\Lambda} |\kappa| \frac{v_\kappa^{(1)}(\omega, \lambda)}{u_\kappa^{(1)}(\omega, \lambda)} \leqslant -\frac{1}{2}\alpha_* \qquad \text{if} \quad \alpha_* := \inf_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{a(x, \lambda)}{b(x, \lambda)} > 0,$$

$$\liminf_{\kappa\to+\infty} \inf_{\lambda\in\Lambda} |\kappa| \frac{v_\kappa^{(1)}(\omega, \lambda)}{u_\kappa^{(1)}(\omega, \lambda)} \geqslant -\frac{1}{2}\alpha^* \qquad \text{if} \quad \alpha^* := \sup_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{a(x, \lambda)}{b(x, \lambda)} < 0.$$

(b) The functions $u_\kappa^{(2)}, v_\kappa^{(2)}$ *are continuous on* $\Omega \times \Lambda$, $v_\kappa^{(2)}(x, \lambda) > 0$ *and*

$$v_\kappa^{(2)}(x, \lambda) \begin{cases} \geqslant \exp\left(\sqrt{\kappa^2 - \alpha\gamma} \int_x^\omega b(t, \lambda)\, \mathrm{d}t\right) & \text{if} \quad \kappa \in [\beta, \infty), \\ \leqslant \exp\left(-\sqrt{\kappa^2 - \alpha\gamma} \int_x^\omega b(t, \lambda)\, \mathrm{d}t\right) & \text{if} \quad \kappa \in (-\infty, -\beta] \end{cases}$$

for all $(x, \kappa, \lambda) \in \Omega \times I \times \Lambda$. *In addition,*

$$\sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{u_\kappa^{(2)}(x, \lambda)}{v_\kappa^{(2)}(x, \lambda)} \right| \leqslant \gamma$$

for all $\kappa \in I$, $u_\kappa^{(2)}(\omega, \cdot) \equiv 0$ *on* Λ *for all* $\kappa \in [\beta, \infty)$, *and*

$$\liminf_{\kappa\to-\infty} \inf_{\lambda\in\Lambda} |\kappa| \frac{u_\kappa^{(2)}(\omega, \lambda)}{v_\kappa^{(2)}(\omega, \lambda)} \geqslant \frac{1}{2}\gamma_* \qquad \text{if} \quad \gamma_* := \inf_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{c(x, \lambda)}{b(x, \lambda)} > 0,$$

$$\limsup_{\kappa\to-\infty} \sup_{\lambda\in\Lambda} |\kappa| \frac{u_\kappa^{(2)}(\omega, \lambda)}{v_\kappa^{(2)}(\omega, \lambda)} \leqslant \frac{1}{2}\gamma^* \qquad \text{if} \quad \gamma^* := \sup_{(x,\lambda)\in[\xi,\omega]\times\Lambda} \frac{c(x, \lambda)}{b(x, \lambda)} < 0.$$

Proof. First we prove (b). For this purpose, consider the family of Riccati equations (2.1), and let $z_\kappa(\cdot, \lambda)$ be the solutions of theorem 2.1. If we define

$$v_\kappa^{(2)}(x, \lambda) := \exp\left(\int_x^\omega a(t, \lambda) z_\kappa(t, \lambda) + \kappa b(t, \lambda)\, \mathrm{d}t\right), \qquad (x, \lambda) \in \Omega \times \Lambda,$$

and $u_\kappa^{(2)} := z_\kappa v_\kappa^{(2)}$ for all $\kappa \in I$, then the functions $u_\kappa^{(2)}, v_\kappa^{(2)}$ are continuous on $\Omega \times \Lambda$, and, by (2.1),

$$\frac{\partial}{\partial x} v_\kappa^{(2)} = -a u_\kappa^{(2)} - \kappa b v_\kappa^{(2)}, \qquad \frac{\partial}{\partial x} u_\kappa^{(2)} = \kappa b u_\kappa^{(2)} + c v_\kappa^{(2)}.$$

Thus,

$$y_\kappa^{(2)}(x,\lambda) := \begin{pmatrix} u_\kappa^{(2)}(x,\lambda) \\ v_\kappa^{(2)}(x,\lambda) \end{pmatrix}, \qquad x \in \Omega,$$

is a nontrivial solution of (3.1) for all $(\kappa, \lambda) \in I \times \Lambda$. Further,

$$\begin{aligned}(\operatorname{sign}\kappa)\,(az_\kappa + \kappa b) &= \left(|\kappa| + (\operatorname{sign}\kappa)\frac{a}{b} z_\kappa\right) b \\ &\geqslant (|\kappa| - \alpha\mu_\kappa) b = \sqrt{\kappa^2 - \alpha\gamma}\, b\end{aligned}$$

implies the first two estimates in (b). Finally, by theorem 2.1,

$$\sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{u_\kappa^{(2)}(x,\lambda)}{v_\kappa^{(2)}(x,\lambda)} \right| \leqslant |\kappa|\mu_\kappa \leqslant \gamma$$

for all $\kappa \in I$, $u_\kappa^{(2)}(\omega, \cdot) \equiv 0$ on Λ for all $\kappa \in [\beta, \infty)$, and the last two estimates in (b) follow from the definition of $u_\kappa^{(2)}$ and from the last two estimates in theorem 2.1.

In order to prove (a), we construct a solution of (3.1) which is linearly independent of $y_\kappa^{(2)}$ by considering the Riccati differential equations

$$w'(x) = c(x,\lambda) w(x)^2 - 2\kappa b(x,\lambda) w(x) + a(x,\lambda), \qquad x \in \Omega. \tag{3.4}$$

Applying theorem 2.1 with a, c exchanged and κ replaced by $-\kappa$, we obtain that (3.4) has solutions $w_\kappa(\cdot, \lambda)$ for all $(\kappa, \lambda) \in I \times \Lambda$ with the properties that w_κ is continuous on $\Omega \times \Lambda$ and bounded by

$$\nu_\kappa := \frac{\alpha}{|\kappa| + \sqrt{\kappa^2 - \alpha\gamma}}$$

for all $\kappa \in I$, $w_\kappa(\omega, \cdot) \equiv 0$ on Λ for all $\kappa \in (-\infty, -\beta]$, and

$$\liminf_{\kappa\to+\infty} \inf_{\lambda\in\Lambda} |\kappa| w_\kappa(\omega,\lambda) \geqslant \tfrac{1}{2}\alpha_* \qquad \text{if} \quad \alpha_* > 0, \tag{3.5}$$

$$\limsup_{\kappa\to+\infty} \sup_{\lambda\in\Lambda} |\kappa| w_\kappa(\omega,\lambda) \leqslant \tfrac{1}{2}\alpha^* \qquad \text{if} \quad \alpha^* < 0. \tag{3.6}$$

If we define

$$u_\kappa^{(1)}(x,\lambda) := \exp\left(\int_x^\omega c(t,\lambda) w_\kappa(t,\lambda) - \kappa b(t,\lambda)\,\mathrm{d}t\right), \qquad (x,\lambda) \in \Omega \times \Lambda,$$

and $v_\kappa^{(1)} := -w_\kappa u_\kappa^{(1)}$ for all $\kappa \in I$, then $u_\kappa^{(1)}, v_\kappa^{(1)}$ are continuous functions on $\Omega \times \Lambda$, and, by (3.4),

$$\frac{\partial}{\partial x} u_\kappa^{(1)} = \kappa b u_\kappa^{(1)} + c v_\kappa^{(1)}, \qquad \frac{\partial}{\partial x} v_\kappa^{(1)} = -a u_\kappa^{(1)} - \kappa b v_\kappa^{(1)}.$$

This implies that

$$y_\kappa^{(1)}(x,\lambda) := \begin{pmatrix} u_\kappa^{(1)}(x,\lambda) \\ v_\kappa^{(1)}(x,\lambda) \end{pmatrix}, \qquad (x,\lambda) \in \Omega \times \Lambda,$$

is also a nontrivial solution of (3.1) for all $(\kappa, \lambda) \in I \times \Lambda$. The first two estimates in (a) follow from

$$(\operatorname{sign}\kappa)(cw_\kappa - \kappa b) = \left(-|\kappa| + (\operatorname{sign}\kappa)\frac{c}{b} w_\kappa\right) b \leqslant (-|\kappa| + \gamma\nu_\kappa) b = -\sqrt{\kappa^2 - \alpha\gamma} b.$$

In addition, by theorem 2.1,

$$\sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{v_\kappa^{(1)}(x,\lambda)}{u_\kappa^{(1)}(x,\lambda)} \right| \leqslant |\kappa|\nu_\kappa \leqslant \alpha$$

for all $\kappa \in I$, $v_\kappa^{(1)}(\omega, \cdot) \equiv 0$ on Λ for all $\kappa \in (-\infty, -\beta]$, and the last two estimates in (a) follow from the definition of $v_\kappa^{(1)}$ and from (3.5) and (3.6).

Finally, defining $Y_\kappa(x, \lambda)$ as in (3.3) and observing that

$$\mu_\kappa \nu_\kappa = \frac{|\kappa| - \sqrt{\kappa^2 - \alpha\gamma}}{|\kappa| + \sqrt{\kappa^2 - \alpha\gamma}} < 1,$$

we conclude that on $\Omega \times \Lambda$

$$\begin{aligned}\det Y_\kappa &= u_\kappa^{(1)} v_\kappa^{(2)}(1 + w_\kappa z_\kappa) \geqslant u_\kappa^{(1)} v_\kappa^{(2)}(1 - |w_\kappa||z_\kappa|) \\ &\geqslant u_\kappa^{(1)} v_\kappa^{(2)}(1 - \mu_\kappa \nu_\kappa) > 0,\end{aligned}$$

and therefore $Y_\kappa(\cdot, \lambda)$ is a fundamental matrix of (3.1) for all $(\kappa, \lambda) \in I \times \Lambda$. □

As a special case, we consider Dirac systems (3.1) with $b(x, \lambda) = \frac{1}{x}$, that is,

$$Jy'(x) + \begin{pmatrix} a(x,\lambda) & \frac{\kappa}{x} \\ \frac{\kappa}{x} & c(x,\lambda) \end{pmatrix} y(x) = 0, \qquad x \in \Omega. \tag{3.7}$$

Corollary 3.2. *Suppose that in (3.7) the functions $a(\cdot, \lambda)$, $c(\cdot, \lambda)$ are measurable on Ω for all $\lambda \in \Lambda$ and the functions $a(x, \cdot)$, $c(x, \cdot)$ are continuous on Λ for all $x \in \Omega$. If*

$$\alpha := \sup_{(x,\lambda)\in\Omega\times\Lambda} |xa(x,\lambda)| < \infty, \qquad \gamma := \sup_{(x,\lambda)\in\Omega\times\Lambda} |xc(x,\lambda)| < \infty,$$

and the estimate $\alpha\gamma < \beta^2 - \frac{1}{4}$ holds, then, for all $(\kappa, \lambda) \in I \times \Lambda$, (3.7) is in the limit point case at $x = 0$. Moreover, the Dirac system (3.7) has a square-integrable solution

$$y_\kappa(x, \lambda) = \begin{pmatrix} u_\kappa(x,\lambda) \\ v_\kappa(x,\lambda) \end{pmatrix}, \qquad x \in \Omega,$$

such that y_κ is continuous on $\Omega \times \Lambda$, where $u_\kappa(x, \lambda) > 0$ if $\kappa \geqslant \beta$ and $v_\kappa(x, \lambda) > 0$ if $\kappa \leqslant -\beta$. In addition,

$$\limsup_{\kappa\to+\infty} \sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{v_\kappa(x,\lambda)}{u_\kappa(x,\lambda)} \right| \leqslant \alpha, \qquad \limsup_{\kappa\to-\infty} \sup_{(x,\lambda)\in\Omega\times\Lambda} \left| \kappa \frac{u_\kappa(x,\lambda)}{v_\kappa(x,\lambda)} \right| \leqslant \gamma,$$

and

$$\liminf_{\kappa\to+\infty} \inf_{\lambda\in\Lambda} |\kappa| \frac{v_\kappa(\omega,\lambda)}{u_\kappa(\omega,\lambda)} > 0 \qquad \text{if} \quad \sup_{(x,\lambda)\in[\xi,\omega]\times\Lambda} xa(x,\lambda) < 0,$$

$$\liminf_{\kappa\to-\infty} \inf_{\lambda\in\Lambda} |\kappa| \frac{u_\kappa(\omega,\lambda)}{v_\kappa(\omega,\lambda)} > 0 \qquad \text{if} \quad \inf_{(x,\lambda)\in[\xi,\omega]\times\Lambda} xc(x,\lambda) > 0$$

with some point $\xi \in (0, \omega)$.

Proof. If we set $b(x, \lambda) := \frac{1}{x}$, $(x, \lambda) \in \Omega \times \Lambda$, then the functions $a, b, c : \Omega \times \Lambda \longrightarrow \mathbb{R}$ satisfy the conditions (i) and (ii) of section 2, and the differential equation (3.7) has the form (3.1). Hence theorem 3.1 can be applied to (3.7). Since

$$\int_x^\omega b(t,\lambda)\,\mathrm{d}t = \log\left(\frac{\omega}{x}\right), \qquad x \in \Omega,$$

we have

$$\exp\left(\pm\sqrt{\kappa^2 - \alpha\gamma} \int_x^\omega b(t,\lambda)\,\mathrm{d}t\right) = \omega^{\pm\sqrt{\kappa^2-\alpha\gamma}} x^{\mp\sqrt{\kappa^2-\alpha\gamma}}, \qquad x \in \Omega.$$

Now let $Y_\kappa(x,\lambda) = \big(y_\kappa^{(1)}(x,\lambda)\; y_\kappa^{(2)}(x,\lambda)\big)$ denote the fundamental matrix of (3.7) obtained from theorem 3.1. The latter and the definitions of $v_\kappa^{(1)}$ and $u_\kappa^{(2)}$ in its proof yield that

$$\big|y_\kappa^{(1)}(x,\lambda)\big| \leqslant C_\kappa x^{\sqrt{\kappa^2-\alpha\gamma}}, \qquad \big|y_\kappa^{(2)}(x,\lambda)\big| \geqslant \widetilde{C}_\kappa x^{-\sqrt{\kappa^2-\alpha\gamma}} \qquad \text{if} \quad \kappa \in [\beta,\infty),$$
$$\big|y_\kappa^{(1)}(x,\lambda)\big| \geqslant \widetilde{C}_\kappa x^{-\sqrt{\kappa^2-\alpha\gamma}}, \qquad \big|y_\kappa^{(2)}(x,\lambda)\big| \leqslant C_\kappa x^{\sqrt{\kappa^2-\alpha\gamma}} \qquad \text{if} \quad \kappa \in (-\infty,-\beta]$$

with some positive constants C_κ and $\widetilde{C}_\kappa$ (here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$). Therefore, since $\sqrt{\kappa^2-\alpha\gamma} > \frac{1}{2}$ by assumption, the square-integrable solutions of (3.7) are constant multiples of the functions

$$y_\kappa(x,\lambda) := \begin{cases} y_\kappa^{(1)}(x,\lambda) & \text{if} \quad \kappa \in [\beta,\infty), \\ y_\kappa^{(2)}(x,\lambda) & \text{if} \quad \kappa \in (-\infty,-\beta], \end{cases}$$

and the properties of $y_\kappa(x,\lambda)$ follow from the results in theorem 3.1. □

Remark 3.3. In particular, corollary 3.2 implies that $v_\kappa(\omega,\lambda) > 0$, $\lambda \in \Lambda$, for sufficiently large $|\kappa|$ and

$$\lim_{\kappa\to-\infty} \sup_{\lambda\in\Lambda} \left|\frac{u_\kappa(\omega,\lambda)}{v_\kappa(\omega,\lambda)}\right| = 0, \qquad \inf_{\lambda\in\Lambda} \frac{u_\kappa(\omega,\lambda)}{v_\kappa(\omega,\lambda)} \to +\infty \qquad \text{for} \quad \kappa \to +\infty,$$

if $a(x,\lambda) \leqslant A < 0$ for all $(x,\lambda) \in [\xi,\omega]\times\Lambda$ with some point $\xi \in (0,\omega)$. Similarly, we have $u_\kappa(\omega,\lambda) > 0$, $\lambda \in \Lambda$, for sufficiently large $|\kappa|$ and

$$\lim_{\kappa\to+\infty} \sup_{\lambda\in\Lambda} \left|\frac{v_\kappa(\omega,\lambda)}{u_\kappa(\omega,\lambda)}\right| = 0, \qquad \inf_{\lambda\in\Lambda} \frac{v_\kappa(\omega,\lambda)}{u_\kappa(\omega,\lambda)} \to +\infty \qquad \text{for} \quad \kappa \to -\infty,$$

provided that $c(x,\lambda) \geqslant C > 0$ for all $(x,\lambda) \in [\xi,\omega]\times\Lambda$.

4. Principal solutions of Dirac systems

In the following, we present a continuity property and a comparison theorem for the principal solutions of (3.1) when κ is fixed. The notion of principal solutions has been introduced first for Sturm–Liouville problems (see, e.g., [5, chapter XI, section 6] or [8, chapter IV, section 3]). A nontrivial solution $y_0 : \Omega \longrightarrow \mathbb{R}^2$ of (3.1),

$$y_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \qquad x \in \Omega,$$

is called *principal* (at $x = 0$), if there exists a real-valued solution y of (3.1),

$$y(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \qquad x \in \Omega,$$

which is linearly independent of y_0, and either of the pair of conditions $v(x) \neq 0$, $\lim_{x\to0} \frac{v_0(x)}{v(x)} = 0$ or $u(x) \neq 0$, $\lim_{x\to0} \frac{u_0(x)}{u(x)} = 0$ holds in a neighbourhood of $x = 0$ (see section 2 in [10]).

In order to specify the principal solutions of (3.1) for fixed κ, we consider the fundamental system of solutions

$$y^{(1)}(x,\lambda) := \begin{pmatrix} u^{(1)}(x,\lambda) \\ v^{(1)}(x,\lambda) \end{pmatrix}, \qquad y^{(2)}(x,\lambda) := \begin{pmatrix} u^{(2)}(x,\lambda) \\ v^{(2)}(x,\lambda) \end{pmatrix}$$

from theorem 3.1, and we define

$$y_0(x,\lambda) := \begin{cases} y^{(1)}(x,\lambda) & \text{if} \quad \kappa > 0, \\ y^{(2)}(x,\lambda) & \text{if} \quad \kappa < 0. \end{cases}$$

Here and in the rest of this section, the index κ will always be omitted.

In addition to the conditions (i) and (ii), we will also need the following assumption on the coefficient b:

(iii) For each $\lambda \in \Lambda$ we have $\int_x^\omega b(t,\lambda)\,\mathrm{d}t \to \infty$ if $x \to 0$.

An immediate consequence of (iii) and theorem 3.1 is:

Proposition 4.1. *If the conditions (i), (ii) and (iii) hold, then the function $y_0(\cdot,\lambda)$ is a principal solution of (3.1) for every $\lambda \in \Lambda$. In addition, for a fixed $\lambda \in \Lambda$, a solution y of (3.1) is principal if and only if $y = Cy_0(\cdot,\lambda)$ with some constant $C \in \mathbb{R}\backslash\{0\}$.*

We can also characterize the principal solutions of (3.1) by the asymptotic behaviour of the Prüfer angles at the origin. If $y : \Omega \longrightarrow \mathbb{R}^2$ is a nontrivial solution of (3.1),

$$y(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \qquad x \in \Omega,$$

then we can write the components of y in polar coordinates:

$$u(x) = \rho(x)\cos\phi(x), \qquad v(x) = \rho(x)\sin\phi(x), \qquad x \in \Omega,$$

with $\rho(x)^2 = u(x)^2 + v(x)^2 \neq 0$ and

$$\phi(x) = \begin{cases} \arctan\frac{v(x)}{u(x)} & \text{if} \quad u(x) \neq 0, \\ \operatorname{arccot}\frac{u(x)}{v(x)} & \text{if} \quad v(x) \neq 0, \end{cases}$$

where the branches of arctan and arccot are chosen such that $\phi : \Omega \longrightarrow \mathbb{R}$ is absolutely continuous. The function ϕ is called *Prüfer angle* (or angle function) of y and it is uniquely defined up to an additive constant $k\pi (k \in \mathbb{Z})$.

Proposition 4.2. *Suppose that the conditions (i), (ii), and (iii) are satisfied. For a fixed $\lambda \in \Lambda$, let y be a nontrivial solution of (3.1). Then every Prüfer angle of y is bounded on Ω. Moreover, y is principal at $x = 0$ if and only if there exists an Prüfer angle ϕ_0 of y such that for all $x \in \Omega$*

$$\phi_0(x) \in \begin{cases} \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) & \textit{if} \quad \kappa > 0, \\ \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) & \textit{if} \quad \kappa < 0. \end{cases} \tag{4.1}$$

Proof. For a fixed $\lambda \in \Lambda$, let $y = y(\cdot,\lambda) : \Omega \longrightarrow \mathbb{R}^2$ be a nontrivial solution of (3.1),

$$y(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \qquad x \in \Omega.$$

Then there exist constants $c_1, c_2 \in \mathbb{R}, |c_1| + |c_2| > 0$, such that $y(x) = c_1 y^{(1)}(x,\lambda) + c_2 y^{(2)}(x,\lambda)$ for all $x \in \Omega$. First we suppose that $\kappa > 0$. If $c_2 = 0$, then y is principal at $x = 0$, and from $|v^{(1)}(x,\lambda)| \leqslant \frac{\alpha}{\kappa} u^{(1)}(x,\lambda)$ it follows that

$$\left|\frac{v(x)}{u(x)}\right| = \left|\frac{v^{(1)}(x,\lambda)}{u^{(1)}(x,\lambda)}\right| \leqslant \frac{\alpha}{\kappa} < 1$$

(note that $u^{(1)}(x,\lambda) > 0$ for all $(x,\lambda) \in \Omega \times \Lambda$). Hence, if we define $\phi_0(x) := \operatorname{Arctan}\frac{v(x)}{u(x)}$, where $\operatorname{Arctan} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ denotes the main branch of the function arctan, then ϕ_0 is an

Prüfer angle of y and $\phi_0(x) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ for all $x \in \Omega$. Now, let $c_2 \neq 0$. Since $v^{(2)}(x,\lambda) > 0$ for all $(x,\lambda) \in \Omega \times \Lambda$ and

$$\lim_{x\to 0}\left|\frac{u^{(1)}(x,\lambda)}{v^{(2)}(x,\lambda)}\right| = \lim_{x\to 0}\left|\frac{v^{(1)}(x,\lambda)}{v^{(2)}(x,\lambda)}\right| = 0, \qquad \sup_{x\in\Omega}\left|\frac{u^{(2)}(x,\lambda)}{v^{(2)}(x,\lambda)}\right| \leqslant \frac{\gamma}{\kappa},$$

we obtain that

$$\limsup_{x\to 0}\left|\frac{u(x)}{v(x)}\right| = \limsup_{x\to 0}\left|\frac{\frac{c_1}{c_2}\frac{u^{(1)}(x,\lambda)}{v^{(2)}(x,\lambda)} + \frac{u^{(2)}(x,\lambda)}{v^{(2)}(x,\lambda)}}{\frac{c_1}{c_2}\frac{v^{(1)}(x,\lambda)}{v^{(2)}(x,\lambda)} + 1}\right| \leqslant \frac{\gamma}{\kappa} < 1. \tag{4.2}$$

Since any Prüfer angle ϕ of y has the form

$$\phi(x) = \operatorname{Arccot}\frac{u(x)}{v(x)} + k\pi,$$

where $\operatorname{Arccot} : \mathbb{R} \longrightarrow (0,\pi)$ is the main branch of arccot and $k \in \mathbb{Z}$, it follows that ϕ is bounded on Ω, and (4.2) implies that $k\pi + \frac{\pi}{4} < \phi(x) < k\pi + \frac{3\pi}{4}$ in a neighbourhood of $x = 0$. In particular, $\phi(x) \notin \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ for sufficiently small $x \in \Omega$. By a similar reasoning, we obtain the assertion for $\kappa < 0$. □

The following result is a comparison theorem (with respect to the parameter λ) for the principal solutions of (3.1).

Theorem 4.3. *Suppose that Q has the form (3.2) and that the conditions (i), (ii) and (iii) are satisfied. Moreover, let $y_0(\cdot,\lambda)$ be a principal solution of (3.1) for every $\lambda \in \Lambda$, and assume that $\phi_0(\cdot,\lambda)$ is the Prüfer angle of $y_0(\cdot,\lambda)$ which satisfies (4.1) for all $x \in \Omega$.*

(a) If $Q(\cdot,\lambda_1) \geqslant Q(\cdot,\lambda_2)$ holds a.e. in Ω for all $\lambda_1 < \lambda_2$ in Λ, then the function $\lambda \longmapsto \phi_0(\omega,\lambda)$ is increasing on Λ.

(b) If $Q(\cdot,\lambda_1) \leqslant Q(\cdot,\lambda_2)$ holds a.e. in Ω for all $\lambda_1 < \lambda_2$ in Λ, then the function $\lambda \longmapsto \phi_0(\omega,\lambda)$ is decreasing on Λ.

Proof. Here, we will verify only (a) in the case $\kappa > 0$; the proof of the remaining assertions is analogous. To this end, we assume to the contrary that $\phi_0(\omega,\lambda_1) > \phi_0(\omega,\lambda_2)$ holds for some $\lambda_1 < \lambda_2$ in Λ. Let

$$\theta := \frac{\phi_0(\omega,\lambda_1) + \phi_0(\omega,\lambda_2)}{2}.$$

If y is the solution of (3.1) for $\lambda = \lambda_1$ which satisfies

$$y(\omega) = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix},$$

then y and $y_0(\cdot,\lambda_1)$ are linearly independent due to the choice of θ. Moreover, if ϕ denotes the Prüfer angle of y with $\phi(\omega) = \theta$, then $\phi_0(\omega,\lambda_1) > \phi(\omega) > \phi_0(\omega,\lambda_2)$. Since $-Q(\cdot,\lambda_1) \leqslant -Q(\cdot,\lambda_2)$ holds a.e. in Ω, we can apply the Comparison theorem 16.1 in [13] which yields $\phi_0(x,\lambda_1) \geqslant \phi(x) \geqslant \phi_0(x,\lambda_2)$ for all $x \in (0,\omega]$. From $\phi_0(x,\lambda_i) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $i \in \{1,2\}$, it follows that $\phi(x) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ for all $x \in \Omega$. Hence, by proposition 4.2, y is a principal solution of (3.1), and proposition 4.1 implies that y is a constant multiple of $y_0(\cdot,\lambda_1)$, a contradiction. □

5. Application to the Dirac operator

In the following, we apply the results of the previous sections to the Dirac operator

$$H = -\mathrm{i}\alpha \cdot \nabla + \alpha_0 + V(|x|)I$$

in $\mathrm{L}^2(\mathbb{R}^3)^4$ with a spherically symmetric potential $V : (0, \infty) \longrightarrow \mathbb{R}$. The units are chosen such that $\hbar = m = c = 1$, I is the 4×4 unit matrix, and

$$\alpha = (\alpha_1, \alpha_2, \alpha_3),$$

where α_k are Hermitian 4×4 matrices satisfying the commutation relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I, \qquad i, j \in \{0, \dots, 3\}.$$

Further, we assume that the potential V satisfies

$$\text{(L)} \qquad V \in \mathrm{L}^\infty_{\mathrm{loc}}(0, \infty), \qquad \lim_{x\to\infty} V(x) = 0, \qquad \limsup_{x\to 0} |xV(x)| < \frac{1}{2}\sqrt{3}.$$

Then, by [11, theorem 1], the operator H is self-adjoint on the domain $\mathcal{D}(H) = \mathrm{H}^1(\mathbb{R}^3)^4$, and

$$\sigma_{\mathrm{ess}}(H) = (-\infty, -1] \cup [1, \infty).$$

Since V is spherically symmetric, there exists an orthogonal decomposition

$$\mathrm{L}^2(\mathbb{R}^3)^4 = \bigoplus_{\kappa\in\mathbb{Z}\setminus\{0\}} \bigoplus_{\ell=1}^{\ell(\kappa)} \mathcal{S}_{\kappa,\ell}$$

which completely reduces H (see [13, section 1]), and the restriction $H \restriction \mathcal{S}_{\kappa,\ell}$ of H to $\mathcal{S}_{\kappa,\ell}$ is unitarily equivalent to the so-called *radial Dirac operator* H_κ (or separated Dirac operator, compare [3]) given by

$$H_\kappa y(x) = Jy'(x) + \begin{pmatrix} -1 + V(x) & \frac{\kappa}{x} \\ \frac{\kappa}{x} & 1 + V(x) \end{pmatrix} y(x), \qquad x \in (0, \infty),$$

and $\mathcal{D}(H_\kappa) = \mathrm{H}^1(0, \infty)^2$. In particular, each H_κ is a self-adjoint operator and

$$H \cong \bigoplus_{\kappa\in\mathbb{Z}\setminus\{0\}} \bigoplus_{\ell=1}^{\ell(\kappa)} H_\kappa.$$

Now, from theorem 16.6 in [13] it follows that $\mathbb{R}\setminus(-1, 1) \subset \sigma_{\mathrm{ess}}(H_\kappa)$, and since $\sigma_{\mathrm{ess}}(H) \cap (-1, 1) = \emptyset$, theorem XIII.85(d) in [9] implies that $\sigma_{\mathrm{ess}}(H_\kappa) \cap (-1, 1) = \emptyset$. Hence, $\sigma_{\mathrm{ess}}(H_\kappa) = (-\infty, -1] \cup [1, \infty)$ is the essential spectrum of the radial Dirac operator H_κ. Moreover, by theorem XIII.85(e) in [9], we have the following relation between the point spectra of H and H_κ:

$$\sigma_{\mathrm{p}}(H) = \bigcup_{\kappa\in\mathbb{Z}\setminus\{0\}} \sigma_{\mathrm{p}}(H_\kappa).$$

This means, a point $\lambda \in \mathbb{R}$ is an eigenvalue of H if and only if there exists an index $\kappa \in \mathbb{Z}\setminus\{0\}$ such that λ is an eigenvalue of H_κ.

Since $\sigma_{\mathrm{ess}}(H) = \mathbb{R}\setminus(-1, 1)$, H has only discrete eigenvalues of finite multiplicity in the gap $(-1, 1)$, and these eigenvalues can accumulate at most at the boundary points ± 1. In the following, we investigate the problem whether ± 1 are accumulation points of eigenvalues of H or not.

Theorem 5.1. *Let* $\lambda_0 \in (-1, 1)$ *and set* $\Lambda := [\lambda_0, 1)$. *If* $\liminf_{x\to\infty} x^2 V(x) > -\infty$, *then* H_κ *has no eigenvalues in* Λ *for sufficiently large* $|\kappa|$.

Proof. A point $\lambda \in (-1, 1)$ is an eigenvalue of H_κ, $\kappa \in \mathbb{Z}\backslash\{0\}$, if and only if the Dirac system

$$Jy'(x) + \begin{pmatrix} V(x) - 1 - \lambda & \frac{\kappa}{x} \\ \frac{\kappa}{x} & V(x) + 1 - \lambda \end{pmatrix} y(x) = 0, \qquad x \in (0, \infty), \tag{5.1}$$

has a nontrivial solution $y \in \mathrm{L}^2(0, \infty)^2$. Now, we fix some $0 < \varepsilon < 1 + \lambda_0$. As $\lim_{x\to\infty} V(x) = 0$ and $\liminf_{x\to\infty} x^2 V(x) > -\infty$, there exist a point $\xi \in (0, \infty)$ and a constant $\eta > 0$ such that $|V(x)| \leqslant \varepsilon$ and $V(x) \geqslant -\frac{\eta}{x^2}$ for all $x \in [\xi, \infty)$. Set $\omega := \xi + 1$. Further, since V is locally bounded on $(0, \infty)$ and $\limsup_{x\to 0} |xV(x)| < \infty$, there exists a constant $\rho > 0$ with the property that $|V(x) \pm 1 - \lambda| \leqslant \frac{\rho}{x}$ for all $x \in \Omega := (0, \omega)$ and $\lambda \in \Lambda$. If we define $a(x, \lambda) := V(x) - 1 - \lambda$, $c(x, \lambda) := V(x) + 1 - \lambda$ and $b(x, \lambda) := \frac{1}{x}$ for $(x, \lambda) \in \Omega \times \Lambda$, then the functions $a, b, c : \Omega \times \Lambda \longrightarrow \mathbb{R}$ satisfy the conditions (i), (ii) and (iii) specified in sections 2 and 4, and the differential equation (5.1) has the form (3.1). In particular,

$$\alpha := \sup_{(x,\lambda)\in\Omega\times\Lambda} |xa(x, \lambda)| \leqslant \rho, \qquad \gamma := \sup_{(x,\lambda)\in\Omega\times\Lambda} |xc(x, \lambda)| \leqslant \rho.$$

With some constant β such that $\beta^2 > \rho^2 + \frac{1}{4}$, corollary 3.2 implies that the Dirac system (5.1) has square-integrable solutions

$$y_\kappa(x, \lambda) = \begin{pmatrix} u_\kappa(x, \lambda) \\ v_\kappa(x, \lambda) \end{pmatrix}, \qquad x \in \Omega,$$

such that y_κ is continuous on $\Omega \times \Lambda$ for all $\kappa \in I := \mathbb{R}\backslash(-\beta, \beta)$, $u_\kappa(x, \lambda) > 0$ if $\kappa \geqslant \beta$ and $v_\kappa(x, \lambda) > 0$ if $\kappa \leqslant -\beta$. Moreover, since $a(x, \lambda) \leqslant \varepsilon - 1 - \lambda_0 < 0$ for all $x \in [\xi, \omega]$, there exists a number $\kappa_1 > 0$ such that $v_\kappa(\omega, \lambda) > 0$ for all $|\kappa| \geqslant \kappa_1$, and

$$\lim_{\kappa\to-\infty} \sup_{\lambda\in\Lambda} \left| \frac{u_\kappa(\omega, \lambda)}{v_\kappa(\omega, \lambda)} \right| = 0, \qquad \inf_{\lambda\in\Lambda} \frac{u_\kappa(\omega, \lambda)}{v_\kappa(\omega, \lambda)} \to +\infty \qquad \text{for} \quad \kappa \to +\infty \tag{5.2}$$

(see remark 3.3). Now, since (5.1) is in the limit point case at $x = 0$ for all $(\kappa, \lambda) \in I \times \Lambda$ by corollary 3.2, a point $\lambda \in \Lambda$ is an eigenvalue of H_κ if and only if (5.1), restricted to $[\omega, \infty)$, has a solution $y \in \mathrm{L}^2[\omega, \infty)^2$ satisfying the interface condition

$$y(\omega) = C y_\kappa(\omega, \lambda) \tag{5.3}$$

with some constant $C \in \mathbb{R}\backslash\{0\}$. In the following, we will reduce the eigenvalue equation for H_κ to a λ-nonlinear Sturm–Liouville problem on the interval $[\omega, \infty)$. For fixed $\lambda \in \Lambda$, by the transformation

$$y(x) = \begin{pmatrix} x^\kappa \widehat{w}(x) \\ x^{-\kappa} w(x) \end{pmatrix}, \qquad x \in [\omega, \infty), \tag{5.4}$$

the system (5.1) on the x-interval $[\omega, \infty)$ is equivalent to the Sturm–Liouville equation

$$(p_\kappa(x, \lambda) w'(x))' - q_\kappa(x, \lambda) w(x) = 0, \qquad x \in [\omega, \infty), \tag{5.5}$$

where

$$p_\kappa(x, \lambda) = \frac{x^{-2\kappa}}{1 + \lambda - V(x)}, \qquad q_\kappa(x, \lambda) = x^{-2\kappa}(1 - \lambda + V(x)), \tag{5.6}$$

and $\widehat{w}(x) = p_\kappa(x, \lambda) w'(x)$. In order to establish the boundary conditions, we write (5.3) in the form

$$\omega^{2\kappa} \frac{p_\kappa(\omega, \lambda) w'(\omega)}{w(\omega)} = \frac{u_\kappa(\omega, \lambda)}{v_\kappa(\omega, \lambda)}. \tag{5.7}$$

Further, from $\lim_{x\to\infty} V(x) = 0$ it follows that $q_\kappa(x,\lambda) > 0$ for sufficiently large x, and lemmas A.1 and A.2 in [10] imply that a solution w of (5.5) satisfies $x^{-\kappa}w, x^{\kappa}\widehat{w} \in L^2[\omega,\infty)$ if and only if w is principal at ∞. Hence, a point $\lambda \in \Lambda$ is an eigenvalue of H_κ if and only if there exists a principal solution $w = w_\kappa(\cdot,\lambda)$ of (5.5) satisfying (5.7). Next, we will establish some bounds on the left-hand side of (5.7). Note that

$$\frac{x^{-2\kappa}}{2+\varepsilon} \leqslant p(x,\lambda) \leqslant \frac{x^{-2\kappa+2}}{1+\lambda_0-\varepsilon} \tag{5.8}$$

and

$$-\eta x^{-2\kappa-2} \leqslant q(x,\lambda) \leqslant (1-\lambda_0+\varepsilon)x^{-2\kappa} \tag{5.9}$$

for all $x \in [\omega,\infty)$ and $(\kappa,\lambda) \in I \times \Lambda$. If we define

$$\rho_\kappa := \kappa - \frac{1}{2} - \sqrt{\left(\kappa - \frac{1}{2}\right)^2 + 1 - (\lambda_0-\varepsilon)^2} = \frac{(\lambda_0-\varepsilon)^2 - 1}{\kappa - \frac{1}{2} + \sqrt{\left(\kappa-\frac{1}{2}\right)^2 + 1 - (\lambda_0-\varepsilon)^2}}$$

and

$$\sigma_\kappa := \kappa + \frac{1}{2} - \sqrt{\left(\kappa + \frac{1}{2}\right)^2 - \eta(2+\varepsilon)} = \frac{\eta(2+\varepsilon)}{\kappa + \frac{1}{2} + \sqrt{\left(\kappa+\frac{1}{2}\right)^2 - \eta(2+\varepsilon)}}$$

for all $|\kappa| \geqslant \kappa_2$ with some constant $\kappa_2 > \frac{1}{2} + \sqrt{\eta(2+\varepsilon)}$, then x^{ρ_κ} is a principal solution of the Euler equation

$$\left(\frac{x^{-2\kappa+2}}{1+\lambda_0-\varepsilon}w'(x)\right)' - (1-\lambda_0+\varepsilon)x^{-2\kappa}w(x) = 0, \qquad x \in [\omega,\infty),$$

and x^{σ_κ} is a principal solution of the Euler equation

$$\left(\frac{x^{-2\kappa}}{2+\varepsilon}w'(x)\right)' + \eta x^{-2\kappa-2}w(x) = 0, \qquad x \in [\omega,\infty).$$

Because of the estimates (5.8) and (5.9), we can apply the comparison theorem [5, chapter XI, corollary 6.5] which yields that a principal solution $w_\kappa(\cdot,\lambda)$ of (5.5) satisfies $w_\kappa(\omega,\lambda) \neq 0$ and

$$\frac{\rho_\kappa}{1+\lambda_0-\varepsilon}\omega^{-2\kappa+1} \leqslant \frac{p_\kappa(\omega,\lambda)w_\kappa'(\omega,\lambda)}{w_\kappa(\omega,\lambda)} \leqslant \frac{\sigma_\kappa}{2+\varepsilon}\omega^{-2\kappa-1}$$

for all $(\kappa,\lambda) \in I \times \Lambda$. Hence,

$$\frac{\omega}{1+\lambda_0-\varepsilon}\rho_\kappa \leqslant \omega^{2\kappa}\frac{p_\kappa(\omega,\lambda)w_\kappa'(\omega,\lambda)}{w_\kappa(\omega,\lambda)} \leqslant \frac{1}{\omega(2+\varepsilon)}\sigma_\kappa$$

for all $(\kappa,\lambda) \in I \times \Lambda$. Since $\lim_{\kappa\to+\infty}\rho_\kappa = \lim_{\kappa\to+\infty}\sigma_\kappa = 0$ and $\rho_\kappa, \sigma_\kappa \to -\infty$ as $\kappa \to -\infty$, we obtain that

$$\lim_{\kappa\to+\infty}\sup_{\lambda\in\Lambda}\left|\omega^{2\kappa}\frac{p_\kappa(\omega,\lambda)w_\kappa'(\omega,\lambda)}{w_\kappa(\omega,\lambda)}\right| = 0$$

and

$$\sup_{\lambda\in\Lambda}\omega^{2\kappa}\frac{p_\kappa(\omega,\lambda)w_\kappa'(\omega,\lambda)}{w_\kappa(\omega,\lambda)} \to -\infty \qquad \text{for} \quad \kappa \to -\infty.$$

Finally, this result and the asymptotic behaviour (5.2) of the right-hand side of (5.7) for $\kappa \to \pm\infty$ imply that the equation in (5.7) cannot hold for any $\lambda \in \Lambda$ if $|\kappa|$ is sufficiently large. □

Theorem 5.2. *For a fixed $\kappa \in \mathbb{Z}\backslash\{0\}$, the discrete eigenvalues of the radial Dirac operator H_κ accumulate at 1 if*

$$\limsup_{x\to\infty} x^2 V(x) < -\tfrac{1}{8}(2\kappa+1)^2,$$

and they do not accumulate at 1 if

$$\liminf_{x\to\infty} x^2 V(x) > -\tfrac{1}{8}(2\kappa+1)^2.$$

Proof. Let $\kappa \in \mathbb{Z}\backslash\{0\}$ be fixed and set $\Lambda := [0, 1)$. Since $\limsup_{x\to 0} |xV(x)| < \frac{1}{2}\sqrt{3}$, there exist a point $\omega \in (0,\infty)$ and a constant $0 < \rho < \frac{1}{2}\sqrt{3}$ such that $|V(x) \pm 1 - \lambda| \leqslant \frac{\rho}{x}$ for all $x \in \Omega := (0, \omega]$ and $\lambda \in \overline{\Lambda} = [0, 1]$. If we define $a(x, \lambda) := V(x) - 1 - \lambda$, $c(x, \lambda) := V(x) + 1 - \lambda$ and $b(x, \lambda) := \frac{1}{x}$ for $(x, \lambda) \in \Omega \times \overline{\Lambda}$, then the functions $a, b, c : \Omega \times \overline{\Lambda} \longrightarrow \mathbb{R}$ satisfy the conditions (i), (ii) and (iii) specified in sections 2 and 4, and the differential equation (5.1) has the form (3.1). In particular,

$$\alpha := \sup_{(x,\lambda)\in\Omega\times\overline{\Lambda}} |xa(x, \lambda)| \leqslant \rho, \qquad \gamma := \sup_{(x,\lambda)\in\Omega\times\overline{\Lambda}} |xc(x, \lambda)| \leqslant \rho,$$

and $\rho^2 < \frac{3}{4} \leqslant \kappa^2 - \frac{1}{4}$. Now, from corollary 3.2 and proposition 4.1 it follows that the Dirac system (5.1) has square-integrable principal solutions

$$y_0(x, \lambda) = \begin{pmatrix} u_0(x, \lambda) \\ v_0(x, \lambda) \end{pmatrix}, \qquad x \in \Omega,$$

such that y_0 is continuous on $\Omega \times \overline{\Lambda}$. Moreover, if $\phi_0(\cdot, \lambda)$ is the Prüfer angle of $y_0(\cdot, \lambda)$ which satisfies (4.1) for all $(x, \lambda) \in \Omega \times \overline{\Lambda}$, then theorem 4.3 implies that the function $\lambda \longmapsto \phi_0(\omega, \lambda)$ is monotonically increasing on the interval Λ. By the existence and uniqueness theorem (see [13, theorem 2.1], for example), we can extend the solution y_0 of (5.1) and its Prüfer angle ϕ_0 continuously to $(0,\infty) \times \overline{\Lambda}$, and the comparison theorem 16.1 in [13] yields that the Prüfer angle $\phi_0(x, \cdot)$ is increasing on Λ for every $x \in [\omega, \infty)$.

Now, as $\lim_{x\to\infty} V(x) = 0$, there exists a point $\xi \in (\omega, \infty)$ such that $|V(x)| < 1$ for all $x \in [\xi, \infty)$. Note that (5.1) is in the limit point case at $x = 0$ for all $\lambda \in \Lambda$. Hence, for any $\tau \in [\xi, \infty)$, a point $\lambda \in \Lambda$ is an eigenvalue of H_κ if and only if (5.1), restricted to $[\tau, \infty)$, has a solution $y \in \mathrm{L}^2[\tau, \infty)^2$ satisfying the interface condition

$$y(\tau) = Cy_0(\tau, \lambda) \tag{5.10}$$

with some constant $C \in \mathbb{R}\backslash\{0\}$. As in the proof of theorem 5.1 we will reduce the eigenvalue equation for H_κ to a λ-nonlinear Sturm–Liouville problem on the interval $[\tau, \infty)$. By the transformation (5.4), the system (5.1) on the x-interval $[\tau, \infty)$ is equivalent to the Sturm–Liouville equation (5.5) with coefficients (5.6) (note that $1 + \lambda - V(x) \geqslant 1 - V(x) > 0$ for all $x \geqslant \tau$ and $\lambda \in \overline{\Lambda}$). Further, if we define

$$\alpha(\lambda) := \tau^{-\kappa} u_0(\tau, \lambda), \qquad \beta(\lambda) := -\tau^{\kappa} v_0(\tau, \lambda),$$

then the interface condition (5.10) is equivalent to

$$\alpha(\lambda)w(\tau) + \beta(\lambda)\widehat{w}(\tau) = 0. \tag{5.11}$$

Now, from lemmas A.1 and A.2 in [10] it follows that a solution w of (5.5) satisfies $x^{-\kappa}w, x^{\kappa}\widehat{w} \in \mathrm{L}^2[\tau, \infty)$ if and only if w is principal at ∞. Therefore, a point $\lambda \in \Lambda$ is an eigenvalue of H_κ if and only if there exists a principal solution w of (5.5) satisfying (5.11). Hence, the eigenvalues of H_κ in Λ coincide with the eigenvalues of the λ-nonlinear Sturm–Liouville problem (5.5) and (5.11). Such λ-nonlinear boundary value problems have been considered in [10], and in order to apply the results therein, we need to verify the conditions (i)–(iv) and (P), (M) specified in section 4 of [10].

Obviously, $p_\kappa > 0$, and the functions p_κ^{-1}, q_κ are continuous on $[\tau, \infty) \times \overline{\Lambda}$, which shows (i) and (ii). Moreover, there exists a continuous function $\zeta : \Lambda \longrightarrow (\tau, \infty)$ such that $|V| \leqslant \frac{1}{2}(1-\lambda)$ on $[\zeta(\lambda), \infty)$ for all $\lambda \in \Lambda$, and we obtain the estimates

$$\frac{1}{2}x^{2\kappa} \leqslant \frac{1}{p_\kappa(x,\lambda)} \leqslant 2x^{2\kappa}, \qquad \frac{1}{2}(1-\lambda)x^{-2\kappa} \leqslant q_\kappa(x,\lambda) \leqslant \frac{3}{2}(1-\lambda)x^{-2\kappa}$$

for $x \in [\zeta(\lambda), \infty)$ and $\lambda \in \Lambda$. Hence, by theorem 4.4 in [10], the conditions (iii) and (P) are satisfied. Additionally, the functions $\alpha, \beta : \overline{\Lambda} \longrightarrow \mathbb{R}$ are continuous, and since $\underline{y}_0(\tau, \lambda) \neq 0$ by the existence and uniqueness theorem, we have $|\alpha(\lambda)| + |\beta(\lambda)| \neq 0$ for all $\lambda \in \overline{\Lambda}$. Since the function $x^{-\kappa}v_0(\cdot, \lambda)$ is a nontrivial solution of (5.5) on $[\xi, \tau]$, it has no accumulation points of zeros in this compact interval according to the separation theorem. Hence, we can assume that $v_0(\tau, 1) \neq 0$ (otherwise, replace τ by a point in $[\xi, \tau]$ with this property). Now, as $v_0(\tau, \lambda)$ depends continuously on $\lambda \in \Lambda$, we can find a point $\mu \in \Lambda$ such that $v_0(\tau, \lambda) > 0$ and therefore $\beta(\lambda) \neq 0$ for all $\lambda \in [\mu, 1]$. Hence, condition (iv) is satisfied. It remains to verify (M). Since

$$\phi_0(\tau, \lambda) = \operatorname{Arccot}\frac{u_0(\tau, \lambda)}{v_0(\tau, \lambda)} + k\pi$$

with some constant $k = k(\lambda) \in \mathbb{Z}$ and $\phi_0(\tau, \cdot)$ is monotonically increasing on $[\mu, 1)$, we obtain that the mapping

$$\lambda \longmapsto \frac{\alpha(\lambda)}{\beta(\lambda)} = -\tau^{-2\kappa} \cot \phi_0(\tau, \lambda)$$

is also increasing on $[\mu, 1)$. Moreover, $p_\kappa(x, \cdot)$ and $q_\kappa(x, \cdot)$ are decreasing with respect to λ for each $x \in [\tau, \infty)$, and therefore the monotonicity condition (M) is satisfied on $[\mu, 1)$.

Now, corollary 4.1 in [10] yields that the eigenvalues of the radial Dirac operator H_κ in the interval $[\mu, 1)$ accumulate at 1 if and only if

$$\left(\frac{x^{-2\kappa}w'(x)}{2 - V(x)}\right)' - x^{-2\kappa}V(x)w(x) = 0 \tag{5.12}$$

is oscillatory at ∞. Further, we can apply Sturm's comparison theorem to (5.12) and the Euler equation

$$(x^{\gamma+1}w'(x))' - \eta x^{\gamma-1}w(x) = 0. \tag{5.13}$$

which is oscillatory if $\eta < -\frac{1}{4}\gamma^2$ and non-oscillatory if $\eta > -\frac{1}{4}\gamma^2$ (note that $x^{-\frac{1}{2}\gamma \pm \sqrt{\eta + \frac{1}{4}\gamma^2}}$ are fundamental solutions of (5.13)). Hence, if

$$\limsup_{x\to\infty} x^2 V(x) < -\tfrac{1}{8}(2\kappa+1)^2,$$

then (5.12) is oscillatory at ∞, and if

$$\liminf_{x\to\infty} x^2 V(x) > -\tfrac{1}{8}(2\kappa+1)^2,$$

then (5.12) is non-oscillatory, which completes the proof of theorem 5.2. □

Remark 5.3. Theorem 5.2 was proved in [10] under the additional assumption that $\int_0^1 |V(x) - \frac{\rho}{x}|\,dx < \infty$ with some $\rho \in \left[0, \frac{1}{2}\sqrt{3}\right)$. Under stronger assumptions like continuous differentiability and boundedness at 0 of the potential it was proved before by [4]. The condition $V \in L^\infty_{\text{loc}}(0, \infty)$ in (L) is needed in the following theorem on the whole Dirac operator H, in contrast to all other above-mentioned papers where only the radial Dirac operators H_κ are studied.

Theorem 5.4. *Suppose that the potential V fulfils the assumption (L). Then the eigenvalues of the Dirac operator H in* $(-1, 1)$ *accumulate at 1 if*

$$\limsup_{x\to\infty} x^2 V(x) < -\tfrac{1}{8},$$

and they do not accumulate at 1 if

$$\liminf_{x\to\infty} x^2 V(x) > -\tfrac{1}{8}.$$

Proof. Note that 1 is an accumulation point of eigenvalues for H if there exists at least one H_κ, $\kappa \in \mathbb{Z}\backslash\{0\}$, such that 1 is an accumulation point of eigenvalues for H_κ. Moreover, 1 is no accumulation point of eigenvalues for H if at most finitely many H_κ have at most finitely many eigenvalues in $(0, 1)$. Now the assertions follow from theorems 5.2 and 5.1. □

By a similar reasoning, reducing (5.1) to a Sturm–Liouville equation for the first component in (5.4), we obtain analogous results concerning the accumulation of eigenvalues of H at -1:

Theorem 5.5. *Suppose that the potential V fulfils the hypothesis (L). Then the eigenvalues of the Dirac operator H in* $(-1, 1)$ *accumulate at* -1 *if*

$$\liminf_{x\to\infty} x^2 V(x) > \tfrac{1}{8},$$

and they do not accumulate at -1 *if*

$$\limsup_{x\to\infty} x^2 V(x) < \tfrac{1}{8}.$$

Remark 5.6. It is well known (compare [12, theorem 10.37]) that -1 is not an accumulation point of eigenvalues of the Dirac operator H if the potential V is non-positive, i.e., $V(x) \leqslant 0$ for $x \in (0, \infty)$. Theorem 5.4 reproves this result since in this case $\limsup_{x\to\infty} x^2 V(x) < \frac{1}{8}$. In [6] the example of a potential $V(x) = -C/(1 + x^2)$ with some positive constant C is considered, and it is proved that the gap $(-1, 1)$ contains infinitely many eigenvalues if $C > \frac{1}{8}$ but only a finite number if $0 < C < \frac{1}{8}$. The same result can be obtained by theorems 5.4 and 5.5 observing that $\lim_{x\to\infty} x^2 V(x) = -C$.

Acknowledgments

The authors gratefully acknowledge the support of the British Engineering and Physical Sciences Research Council, EPSRC, grant no GR/R40753 and of the German Research Foundation, DFG, grant no TR368/4-1.

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